

Tutoring -

February 22, 2016 7:26 PM

- ① Monotone function w/ dense image is continuous.
- ② $F : C[0,1] \rightarrow \mathbb{R}$ is continuous
 $f \mapsto \int f(t) dt$
& functions with mean value zero are a closed subspace
- ③ If $f: X \rightarrow Y$ is proper, cont. and invertible, then f^{-1} is continuous.
- ④ $A = \{(r \cos e^{\frac{1}{1-r}}, r \sin e^{\frac{1}{1-r}}) : r \in (0,1)\}$ Find \overline{A} .

Problem 2 $(X, d) = (C[0,1], \sup)$

Fix $\epsilon > 0$ and $f \in C[0,1]$. For $g \in C[0,1]$, we estimate:

$$\begin{aligned} \left| \int_0^1 f(t) dt - \int_0^1 g(t) dt \right| &= \left| \int_0^1 f(t) - g(t) dt \right| \quad (\text{linearity of } \int) \\ &\leq \int_0^1 |f(t) - g(t)| dt \quad (\text{triangle ineq. for } \int) \end{aligned}$$

$$= \int_0^1 |f(t) - g(t)| dt \quad (\text{triangle ineq. for } \int)$$

$$\leq \int_0^1 \sup_{t_0 \in [0,1]} |f(t_0) - g(t_0)| dt \quad (\text{monotonicity of } \int)$$

$$< \delta \int_0^1 dt$$

$$< \varepsilon \quad \text{Choose } \delta = d(f, g) < \varepsilon$$

Therefore $F: C[0,1] \rightarrow \mathbb{R}$ is continuous.

$$f \mapsto \int_0^1 f(t) dt$$

Notice that $f_{av} = \frac{1}{1-0} \int_0^1 f(t) dt$. The set

$$\tilde{X} = \{f \in C[0,1] \mid f_{av} = 0\} = F^{-1}(0)$$

$$\lambda = \{ f \in C[0,1] \mid f_{\text{av}} = 0 \} = \{ f(0) \}$$

is closed because it is the preimage of a closed set (points in \mathbb{R} are closed)

③ Prop: If $f: X \rightarrow Y$ is proper, continuous, and invertible then f^{-1} is continuous.

Defn: $f: X \rightarrow Y$ is proper $\Leftrightarrow \forall K \subseteq Y$ compact, $f^{-1}(K)$ is compact.

Proof of prop:

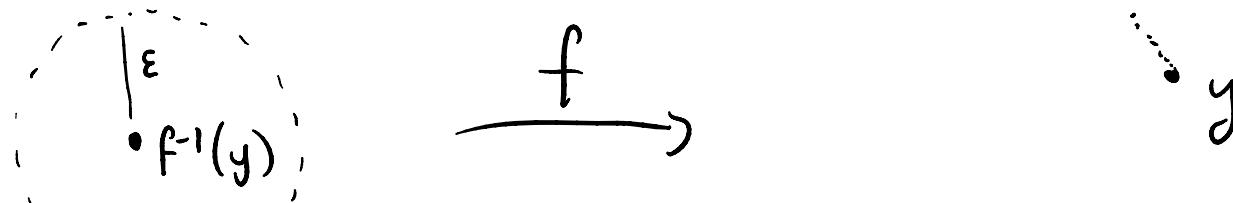
Suppose not, then $\exists y \in Y \ \exists \varepsilon > 0 \ \forall \delta > 0 \ \exists y' \in Y$

s.t. $d_Y(y, y') < \delta$ and $d_X(f^{-1}(y), f^{-1}(y')) \geq \varepsilon$.

Set $\delta = \frac{1}{j}$ and choose y_j s.t. $d_Y(y, y_j) < \frac{1}{j}$ and $d_X(f^{-1}(y), f^{-1}(y_j)) \geq \varepsilon$

Clearly $y_j \rightarrow y$.

... $\{y_n\}$



The set $K = \{y_1, y_1, y_2, y_3, \dots\}$ is compact, so $f^{-1}(K)$ is compact.

The sequence $\{f^{-1}(y_j)\}_{j=1}^{\infty} \subseteq X$ has a convergent subsequence

$$\begin{aligned} \text{and } f\left(\lim_{n \rightarrow \infty} f^{-1}(y_{j_n})\right) &= \lim_{n \rightarrow \infty} f(f^{-1}(y_{j_n})) \\ &= \lim_{n \rightarrow \infty} y_{j_n} \\ &= y \end{aligned}$$

Since f is invertible,

$$\lim_{n \rightarrow \infty} f^{-1}(y_{j_n}) = f^{-1}(y)$$

However, this is impossible since $d(f^{-1}(y_n), f^{-1}(y)) \geq \varepsilon \quad \forall n$.

L

Exercise: Modify the argument above to give a direct proof.

(4) Notice that $f: (0,1) \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$

$$r \mapsto \frac{1}{1-r} \qquad t \mapsto e^t$$

are monotone increasing functions and:

$$(0,1) \xrightarrow{f} (1,\infty) \xrightarrow{g} (e,\infty). \text{ let } F = g \circ f$$

Claim $\overline{A} = \{(0,0)\} \cup A \cup \{x^2+y^2=1\}$

Proof • Clearly $A \subseteq \overline{A}$

• Choosing say, $r_n = \frac{1}{n}$, notice

$$\left\| \left(-\cos F\left(\frac{1}{n}\right), \frac{1}{n} \sin F\left(\frac{1}{n}\right) \right) \right\| = \sqrt{\frac{2}{2}} = \sqrt{2} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\left\| \left(\frac{1}{n} \cos F\left(\frac{1}{n}\right), \frac{1}{n} \sin F\left(\frac{1}{n}\right) \right) \right\| = \sqrt{\frac{c}{n^2}} = \frac{\sqrt{c}}{n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

So $(0,0) \in \bar{A}$

- Finally, a point on the unit circle $(\cos\theta, \sin\theta)$.

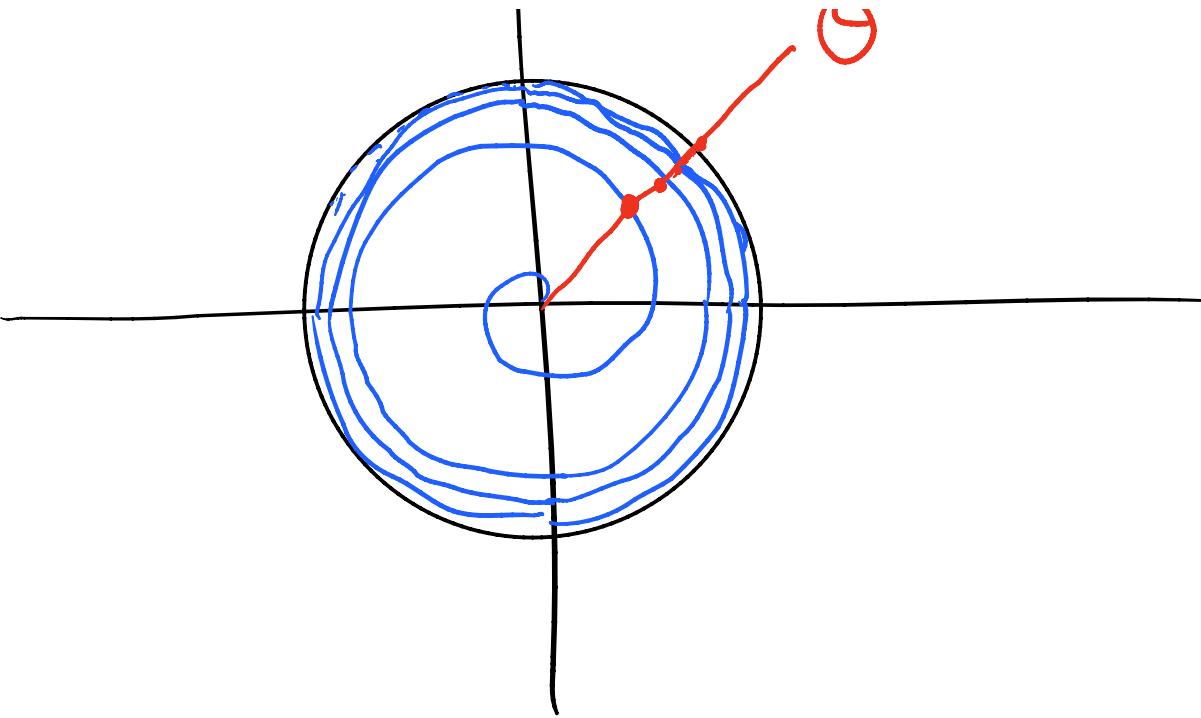
Important step

For every $n \geq 1$, choose the unique $r_n \in (0,1)$ such that

$$F(r_n) = \theta + 2\pi n$$

We may do this because monotone increasing functions are injective.





The r_n have the property that $F(r_n) \rightarrow \infty$ so $r_n \rightarrow 1$

$$\text{Then } \left\| (r_n \cos F(r_n), r_n \sin F(r_n)) - (\cos \theta, \sin \theta) \right\|$$

$$= \left\| ((r_n-1) \cos \theta, (r_n-1) \sin \theta) \right\|$$

$$= \sqrt{(r_n-1)^2 \cos^2 \theta + (r_n-1)^2 \sin^2 \theta}$$

$$= |r_n-1| \longrightarrow 0 \quad \text{as } n \rightarrow \infty$$

So any point of the form $(\cos\theta, \sin\theta) \in \overline{A}$.

Conversely, we must show $\overline{A} \subseteq A$.

• Since $A \subseteq B(1,0)$ we must have $\overline{A} \subseteq \overline{B(1,0)}$

If $(x,y) \neq (0,0)$, and $(x,y) \notin A$, and $(x,y) \notin \{x^2+y^2=1\}$

then we claim $(x,y) \in \text{exterior } A$, and therefore not in \overline{A} .

This is intuitively clear, but awkward to prove.

